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# Remarks on "Floating Bodies Subject to Capillary Attractions"

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**Abstract.** Using a theorem of E. Gutkin from the theory of billiards, we characterize the contact angles $\gamma$ for which there exist horizontal cylinders with non-circular section that can float in any orientation at angle $\gamma$.

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**Keywords.** Capillarity, contact angle, floating criteria.

The reference [1] addressed the question of characterizing the smooth rigid convex cylinders that can float with generators horizontal, on a horizontal liquid surface in zero gravity. For the particular contact angle $\gamma = \pi/2$, [1] contains a procedure for constructing such cylinders with sections that are distinct from circles. For $\gamma \neq \pi/2$, the analogous procedure yielded sections with a singular boundary point at which the contact angle cannot be defined, and with a range of only $\pi$ (as distinct from the desired $2\pi$) of orientations at which floating is feasible.

I am indebted to Sergei Tabachnikov for pointing out to me that the requirement is equivalent to that of finding a smooth convex non-circular billiard table with an invariant curve $\gamma =$ const., and directing me to a discovery of Eugene Gutkin [2] that *such a table exists if and only if the value* $\gamma \in (0, \pi)$ *satisfies*

$$\tan n\gamma = n \tan \gamma \qquad (*)$$

*for some integer* $n > 1$.

One sees easily that for $n = 2$ or 3 there is no solution of $(*)$ in the range of interest $0 < \gamma < \pi$. For all larger $n$, solutions in that range do appear. Figure 1 shows the case $n = 6$ in the half-interval $0 < \gamma < \pi/2$; this case is representative for what occurs, and one infers that there exist $2[(n-2)/2]$ distinct solutions of $(*)$, in $0 < \gamma < \pi$. Here [ ] means "greatest integer equal or less than". Some solutions conceivably coincide with solutions that appear for other values of $n$. It is thus clear that the set of all solutions is both non-null and countable; we conclude *there is a non-null but countable set of* $\gamma$*-values, for which nonsingular sections distinct from circles exist that lead to floating in every orientation.*

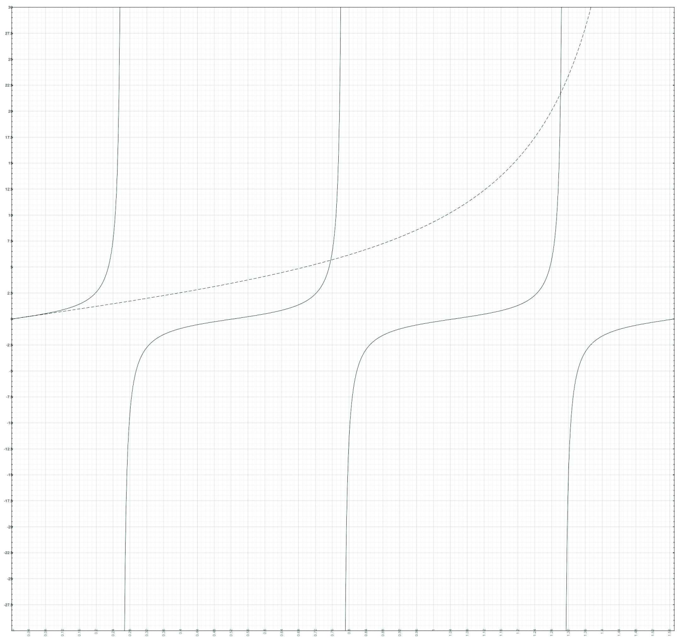

FIGURE 1. The functions $6\tan\gamma$ and $\tan 6\gamma$ in the interval $0 < \gamma < \pi/2$.

A particular consequence is that the set of such $\gamma$ cannot be dense on any subinterval, and thus it should be no surprise that a singularity appeared in the construction offered in [1], for the particular $\gamma \neq \pi/2$ that was chosen there. It remains open to determine whether the procedure indicated in that reference will lead to a non-singular section for one of the countable set of data for which such sections exist.

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## References

[1] R. Finn, Floating Bodies Subject to Capillary Attractions *J. Math. Fluid Mech.* **11** (2009), 443–458.

[2] E. Gutkin, Billiard tables of constant width and dynamical characteristics of the circle, Penn. State Workshop Proceedings, Oct. 1993. An outline appears on pp. 53–54 of the monograph *Billiards* by Serge Tabachnikov, Panoramas et Synthéses I, Soc. Math. France, 1995.

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